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Symmetries, conservation laws and Hamiltonian structures of the non-isospectral and variable coefficient KdV and MKdV equations

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Abstract. An infinite number of form-invariant symmetries is obtained and a one-to-one correspondence between symmetries and conservation laws is established for the non-isospectral and variable coefficient generalizations of both the KdV and the MKdV equations. Two families of symmetries and their Lie algebraic structures are constructed. Some interesting facts about their Hamiltonian structures are presented.

1. Introduction

In [1] we studied the non-isospectral and variable coefficient KdV equation (h-t-KdV) by the use of Bäcklund transformation to construct explicit solutions including solitons with unusual dynamics. The initial value problem of this equation was solved by the inverse scattering method [2]. Non-propagating solitons with their decompositions and interactions were presented. They provide mathematical models for oscillating and standing solitons observed experimentally in [3].

The aim of this article is to investigate the symmmetries, conservation laws and Hamiltonian structures of both the h-t-KdV and the h-t-λ-MKdV equations. It is well know that a necessary condition for the integrability of these equations is the existence of an infinite number of symmetries and the conservation laws contain a lot of important knowledge about the equation under consideration. For the h-t-KdV equation, it is shown that there exist an infinite number of form-invariant symmetries. A one-to-one correspondence between conservation laws and symmetries is established explicitly. Using the Miura transformation, the same conclusions is shown to be true for the h-t-λ-MKdV equation. Furthermore, some Lie algebraic structures of these symmetries are investigated. Lastly, we obtain the second Hamiltonian structure (w.r.t. the standard KdV) of the h-t-KdV equation and the first Hamiltonian structure of the h-t-λ-MKdV equation. It is interesting that the bi-Hamiltonian structures here are elusive and they are replaced by some 'hidden' Hamiltonian structures.

Specifically, we are interested in the following h-t-KdV equation:

$$P(u) = 0 \tag{1}$$

where

$$P(u) = u_t + k_0(u_{xxx} + 6uu_x) + 4k_1u_x - h(2u + xu_x)$$

and the following h-t-λ-MKdV equation:

$$Q(v) = 0 \tag{2}$$

where

$$Q(v) = v_t + k_0(v_{xxx} + 6(\lambda - v^2)v_x) + 4k_1 v_x - h(v + xv_x).$$

In the above equations, k_0, k_1, h and λ are arbitrary functions of t, and λ satisfies the non-isospectral condition

$$\lambda_t = 2h\lambda. \tag{3}$$

Under the Miura transformation

$$u = \lambda - v_x - v^2 \tag{4}$$

the following equality is obtained:

$$P(u) = (-\partial_x - 2v)Q(v). \tag{5}$$

In what follows we shall employ the following notation:

$$\partial_{ix} = \frac{\partial^i}{\partial x^i}$$

$$\partial_x^{-1} = \int_{-\infty}^x$$

$$\delta_{w(x)} = \sum_{i=1}^{\infty}(-1)^i \frac{\partial^i}{\partial x^i}\frac{\partial}{\partial w_{ix}} \qquad (w_{ix} = \partial_{ix} w).$$

2. Symmetries

The symmetry ζ of (1) satisfies the equation

$$P_u(\zeta) = 0 \tag{6}$$

where

$$P_u(\zeta) = \zeta_t + k_0\zeta_{xxx} + (6k_0 u + 4k_1 - hx)\zeta_x + (6k_0 u_x - 2h)\zeta.$$

The symmetry τ of (2) satisfies the equation

$$Q_v(\tau) = 0 \tag{7}$$

where

$$Q_v(\tau) = \tau_t + k_0\tau_{xxx} + (6k_0(\lambda - v^2) + 4k_0 - hx)\tau_x - (12k_0 v v_x + h)\tau.$$

Theorem 2.1.

$$P_u(\zeta) = (-\partial_x - 2v)Q_v(\tau) \tag{8}$$

where

$$\zeta = (-\partial_x - 2v)\tau. \tag{9}$$

Proof. Since

$$Q(v) = 0$$
$$\zeta_t = -\tau_{xt} - 2v_t\tau - 2v\tau_t$$
$$\zeta_x = -\tau_{xx} - 2v_x\tau - 2v\tau_x$$
$$\zeta_{3x} = -\tau_{3x} - 2v_{3x}\tau - 6v_{xx}\tau_x - 6v_x\tau_{xx} - 2v\tau_{3x}$$

the theorem can be proved in substituting them into the expression of $P_u(\zeta)$.

Corollary. There exists a one-to-one correspondence of symmetries between (1) and (2).

We now construct recursion formulas of symmetries of (1) and (2). Assuming that $\zeta_0, \zeta_1, \ldots, \zeta_n$ are symmetries of (1), let

$$\zeta_{n+1} = \exp\left(-2\int h\right)(a_n u\zeta_n + b_n u_x \partial_x^{-1}\zeta_n + c_n\zeta_{n,xx})$$

with $\zeta_0 = \exp\left(-\int h\right)u_x$. We shall determine constants a_n, b_n and c_n in order for ζ_{n+1} to satisfy (6) if ζ_n does.

Under the boundary conditions

$$\zeta_{n,lx} \to 0 \qquad (x \to \pm\infty,\ n \geqslant 0,\ l = 0, 1, \ldots)$$

we have

$$\partial_x^{-1}\zeta_{n,t} + k_0\zeta_{n,xx} + (6k_0u + 4k_1 - hx)\zeta_n - h\partial_x^{-1}\zeta_n = 0$$

$$\zeta_{n,xt} + k_0\zeta_{n,4x} + (6k_0u + 4k_1 - hx)\zeta_{n,xx} + (12k_0u_x - 3h)\zeta_{n,x} + 6k_0u_{xx}\zeta_n = 0$$

$$\zeta_{n,2xt} + k_0\zeta_{n,5x} + (6k_0u + 4k_1 - hx)\zeta_{n,3x} + (18k_0u_x - 4h)\zeta_{n,xx} + 18k_0u_{xx}\zeta_{n,x} + 6k_0u_{3x}\zeta_n = 0$$

$$u_{xt} + k_0u_{4x} + (6k_0u + 4k_1 - hx)u_{xx} + (6k_0u_x - 3h)u_x = 0.$$

Since

$$\zeta_{n+1,t} = -2h\zeta_{n+1} + \exp\left(-2\int h\right)\left[a_n(u_t\zeta_n + u\zeta_{n,t}) + b_n(u_{xt}\partial_x^{-1}\zeta_n + u_x\partial_x^{-1}\zeta_{n,t}) + c_n\zeta_{n,2xt}\right]$$

$$\zeta_{n+1,x} = \exp\left(-2\int h\right)\left[a_n(u_x\zeta_n + u\zeta_{n,x}) + b_n(u_{xx}\partial_x^{-1}\zeta_n + u_x\zeta_n) + c_n\zeta_{n,3x}\right]$$

$$\zeta_{n+1,xx} = \exp\left(-2\int h\right)\left[a_n(u_{xx}\zeta_n + 2u_x\zeta_{n,x} + u\zeta_{n,xx}) + b_n(u_{3x}\partial_x^{-1}\zeta_n + 2u_{xx}\zeta_n + u_x\zeta_{n,x}) + c_n\zeta_{n,4x}\right]$$

$$\zeta_{n+1,3x} = \exp\left(-2\int h\right)\left[a_n(u_{3x}\zeta_n + 3u_{xx}\zeta_{n,x} + 3u_x\zeta_{n,xx} + u\zeta_{n,3x}) + b_n(u_{4x}\partial_x^{-1}\zeta_n + 3u_{3x}\zeta_n + 3u_{xx}\zeta_{n,x} + u_x\zeta_{n,xx}) + c_n\zeta_{n,5x}\right]$$

therefore

$$P_u(\zeta_{n+1}) = 3k_0\exp\left(-2\int h\right)\left[(b_n - 2c_n)u_{3x}\zeta_n + (a_n + b_n - 6c_n)u_{xx}\zeta_{n,x} + (a_n - 4c_n)u_x\zeta_{n,xx}\right].$$

If we take $a_n=4c_n$, $b_n=2c_n$, and $c_n=1$, we obtain

$$\zeta_{n+1} = \exp\left(-2\int h\right)(\partial_x^2 + 4u + 2u_x\partial_x^{-1})\zeta_n \tag{10}$$

and

$$P_u(\zeta_{n+1}) = 0.$$

That is, the recursion formula (10) yields a sequence of solutions for (6) and it implies the following fact.

Theorem 2.2. The h-t-KdV equation (1) has an infinite number of form-invariant symmetries.

The recursion formula for the symmetries τ_n of the equation (2) also exists and can be derived from (4), (9) and (10).

Since

$$\begin{aligned}\tau_{n+1} &= (-\partial_x - 2v)^{-1}\zeta_{n+1}\\ &= \exp\left(-2\int h\right)(-\partial_x - 2v)^{-1}\left[\partial_x^2 + 4(\lambda - v_x - v^2)\right.\\ &\quad \left.- 2(v_{xx} + 2vv_x)\partial_x^{-1}\right](-\partial_x - 2v)\tau_n\\ &\quad \partial_x^2 v = v\partial_x^2 + 2v_x\partial + v_{xx}\\ &\quad \partial_x v^2 = v^2\partial_x + 2vv_x\\ &\quad \partial_x v_x\partial_x^{-1} v = v_{xx}\partial_x^{-1} v + v_x v.\end{aligned}$$

Furthermore

$$\begin{aligned}&\left[\partial_x^2 + 4(\lambda - v_x - v^2) - 2(v_{xx} + 2vv_x)\partial_x^{-1}\right](-\partial_x - 2v)\\ &\quad = -\partial_x^3 - 2v\partial_x^2 - 4(\lambda - v^2)\partial_x - 8\lambda v + 12vv_x + 8v^3 + 4(v_{xx} + 2vv_x)\partial_x^{-1}v\end{aligned}$$

and

$$\begin{aligned}&(-\partial_x - 2v)\left[\partial_x^2 + 4(\lambda - v^2) - 4v_x\partial_x^{-1}v\right]\\ &\quad = -\partial_x^3 - 2v\partial_x^2 - 4(\lambda - v^2)\partial_x - 8\lambda v + 12vv_x + 8v^3 + 4(v_{xx} + 2vv_x)\partial_x^{-1}v\end{aligned}$$

therefore the left-hand sides of above two equalities are equal, and

$$\tau_{n+1} = \exp\left(-2\int h\right)(\partial_x^2 + 4(\lambda - v^2) - 4v_x\partial_x^{-1}v)\tau_n \tag{11}$$

with $\tau_0 = (-\partial_x - 2v)^{-1}\exp\left(-\int h\right)u_x = \exp\left(-\int h\right)v_x$.

Theorem 2.3. The h-t-λ-MKdV equation (2) has also an infinite number of form-invariant symmetries.

Remark. The operator

$$\Phi_u = \exp\left(-2\int h\right)(\partial_x^2 + 4u + 2u_x\partial_x^{-1}) \tag{12}$$

is a hereditary strong symmetry of (1) and the operator

$$\Phi_v = \exp\left(-2\int h\right)(\partial_x^2 + 4(\lambda - v^2) - 4v_x\partial_x^{-1}v) \tag{13}$$

is a hereditary strong symmetry of (2).

Two families of symmetries $\{\zeta_n\}$, $\{\xi_n\}$ of (1) are

$$\zeta_0 = \exp\left(-\int h\right)u_x, \ldots$$
$$\zeta_n = \Phi_u^n(\zeta_0), \ldots$$
$$\xi_0 = 3k\zeta_0 + \tfrac{1}{2}\exp\left(2\int h\right), \ldots$$
$$\xi_n = 3k\zeta_n + \Phi_u^n\left(\tfrac{1}{2}\exp\left(2\int h\right)\right), \ldots .$$

The corresponding two families of symmetries $\{\tau_n\}$, $\{\sigma_n\}$ of (2) given by (9) are

$$\tau_0 = \exp\left(-\int h\right)v_x, \ldots$$
$$\tau_n = \Phi_v^n(\tau_0), \ldots$$
$$\sigma_0 = 3k\tau_0 + (-\partial_x - 2v)^{-1}\left(\tfrac{1}{2}\exp\left(2\int h\right)\right), \ldots$$
$$\sigma_n = 3k\tau_n + \Phi_v^n\left((-\partial_x - 2v)^{-1}\left(\tfrac{1}{2}\exp\left(2\int h\right)\right)\right), \ldots .$$

where

$$k = -\int\left(k_0\exp\left(3\int h\right)\right).$$

In light of the hereditary recursion operators (12), (13), we can immediately obtain the following result, as has been shown by [11].

Theorem 2.4. $\zeta_m, \xi_n (m, n = 0, 1, 2, \ldots)$ satisfy a Lie algebra

$$[\zeta_m, \zeta_n] = 0$$
$$[\zeta_m, \xi_n] = (2m+1)\zeta_{m+n-1} \qquad (m+n \geqslant 1)$$
$$[\xi_m, \xi_n] = 2(m-n)\xi_{m+n-1} \qquad (m+n \geqslant 1)$$

and $\tau_m, \sigma_n (m, n = 0, 1, 2, \ldots)$ satisfy a Lie algebra

$$[\tau_m, \tau_n] = 0$$
$$[\tau_m, \sigma_n] = (2m+1)\tau_{m+n-1} \qquad (m+n \geqslant 1)$$
$$[\sigma_m, \sigma_n] = 2(m-n)\sigma_{m+n-1} \qquad (m+n \geqslant 1).$$

where

$$[a, b] = a'[b] - b'[a].$$

Remark. Each of the following equations:

KdV, MKdV; GKdV, GMKdV (see [5, 6]); h-t-KdV, h-t-λ-MKdV

has two families of symmetries which satisfy the same Lie algebraic structure. Thus, these might imply that the three KdV equations can be mapped to one another by some non-singular transformations and similarly for the three MKdV equations. We shall address these questions elsewhere.

3. Symmetries and conservation laws

We shall establish a one-to-one correspondence between the symmetries and the conservation laws of both the h-t-KdV and the h-t-λ-MKdV equations.

In the spirit of [4], we prove some lemmas which will be useful in what follows.

Lemma 3.1. For an arbitrary function $l(t,x)$ such as $l(t,x) \to 0$ (sufficiently fast for any fixed t as $x \to \pm\infty$), we have

$$\int l(t,x)\delta_w g(t,w)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int g(t, w+\varepsilon l(t,x))\,\mathrm{d}x\bigg|_{\varepsilon=0}.$$

Proof.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int g(t, w+\varepsilon l(t,x))\,\mathrm{d}x\bigg|_{\varepsilon=0} &= \int\Big(\sum \partial_{w_{ix}} g(t,w)\partial_{ix} l(t,x)\Big)\,\mathrm{d}x\\ &= \int l(t,x)\Big(\sum(-1)^i \partial_{ix}\partial_{w_{ix}} g(t,w)\Big)\,\mathrm{d}x\\ &= \int l(t,x)\delta_w g(t,w)\,\mathrm{d}x.\end{aligned}$$

Lemma 3.2. If w satisfies an evolution equation

$$w_t = L(t,w)$$

then

$$g_t(t,w) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int g(t, w+\varepsilon L(t,x))\,\mathrm{d}x\bigg|_{\varepsilon=0}$$

and

$$\int(\delta_w g_t - (\delta_w g)_t) l(t,x)\,\mathrm{d}x = \int \delta_w g \frac{\mathrm{d}}{\mathrm{d}\varepsilon} L(t, w+\varepsilon l(t,x))\bigg|_{\varepsilon=0}\,\mathrm{d}x.$$

Proof.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon} g(t+\varepsilon, w+\varepsilon L(t,w))\bigg|_{\varepsilon=0} &= \partial_t g(t,w) + \sum \partial_{w_{ix}} g(t,w)\partial_{ix} L(t,w)\\ &= \partial_t g(t,w) + \sum \partial_{w_{ix}} g(t,w)\partial_t\partial_{ix} w\\ &= g_t(t,w)\end{aligned}$$

and

$$\begin{aligned}&\int(\delta_w g_t - (\delta_w g)_t) l(t,x)\,\mathrm{d}x\\ &\quad= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\alpha}\int g(t+\alpha, w+\varepsilon l(t,x)+\alpha L(t, w+\varepsilon l))\,\mathrm{d}x\bigg|_{\varepsilon=0}^{\alpha=0}\\ &\quad= \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int g(t+\alpha, w+\varepsilon l(t,x)+\alpha L(t,w))\,\mathrm{d}x\bigg|_{\alpha=0}^{\varepsilon=0}\end{aligned}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int \Big[\Big(\partial_t g + \sum \partial_{w_{ix}} g \partial_{ix} L(t, w + \varepsilon l) \Big)$$

$$- \Big(\partial_t g + \sum \partial_{w_{ix}} g \partial_{ix} L(t, w) \Big) \Big] \,\mathrm{d}x \Big|_{\varepsilon=0}$$

$$= \int \delta_w g \frac{\mathrm{d}}{\mathrm{d}\varepsilon} L(t, w + \varepsilon l(t, x)) \Big|_{\varepsilon=0} \mathrm{d}x.$$

Lemma 3.3. For any function $f(t, w)$, we have

$$\delta_w \partial_x f(t, w) = 0.$$

Suppose that a function $g(t, w)$ satisfies the boundary condition

$$g(t, w) \to 0$$

(for any fixed t as $x \to \pm\infty$) and if $\delta_w g(t, w) = 0$, then

$$g(t, w) = \partial_x(-X(t, w))$$

for some function $X(t, w)$.

Proof. Take $g(t, w) = \partial_x f(t, w)$ in lemma 3.1 for a proof of the first part. For the second part, one may refer to the proof of lemma 3.2 in [4].

Lemma 3.4. If u is a solution of (1), then the following equality holds

$$\delta_u g_t = (\delta_u g)_t + k_0 \partial_x^3 \delta_u g + (6k_0 u + 4k_1 - hx) \partial_x \delta_u g + h \delta_u g.$$

Proof. Take

$$L(t, u) = -k_0(u_{xxx} + 6uu_x) - 4k_1 u_x + h(2u + xu_x)$$

in lemma 3.2, then

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} L(t, u + \varepsilon l(t, x)) \Big|_{\varepsilon=0} = -k_0 l_{xxx} - (6k_0 u + 4k_1 - hx) l_x - (6k_0 u_x - 2h) l.$$

Therefore

$$\int (\delta_u g_t - (\delta_u g)_t) l(t, x) \,\mathrm{d}x$$

$$= \int \delta_u g(-k_0 l_{xxx} - (6k_0 u + 4k_1 - hx) l_x - (6k_0 u_x - 2h) l) \,\mathrm{d}x$$

$$= \int l(t, x)(k_0 \partial_x^3 \delta_u g + (6k_0 u + 4k_1 - hx) \partial_x \delta_u g + h \delta_u g) \,\mathrm{d}x$$

so that the lemma is proved.

Lemma 3.5. If v is a solution of (2), then the following equality holds

$$\delta_v g_t = (\delta_v g)_t + k_0 \partial_x^3 \delta_v g + (6k_0(\lambda - v^2) + 4k_1 - hx) \partial_x \delta_v g.$$

Proof. Take

$$L(t,v) = -k_0(v_{xxx} + 6(\lambda - v^2)v_x) - 4k_1 v_x + h(v + x v_x)$$

in lemma 3.2, then

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} L(t, v + \varepsilon l(t,x))\Big|_{\varepsilon=0} = -k_0 l_{xxx} - (6k_0(\lambda - v^2) + 4k_1 - hx) l_x - (12k_0 v_x - h) l.$$

Therefore

$$\int (\delta_v g_t - (\delta_v g)_t) l(t,x)\,\mathrm{d}x$$

$$= \int \delta_v g(-k_0 l_{xxx} - (6k_0(\lambda - v^2) + 4k_1 - hx) l_x - (12k_0 v_x - h) l)\,\mathrm{d}x$$

$$= \int l(t,x)(k_0 \partial_x^3 \delta_v g + (6k_0(\lambda - v^2) + 4k_1 - hx)\partial_x \delta_v g)\,\mathrm{d}x$$

so that the lemma is proved.

Now we shall establish a close relationship between the symmetries and the conservation laws in the following two theorems.

Theorem 3.1. Suppose the two functions $\zeta(t,u)$, $T(t,u)$ satisfy

$$\zeta = \exp\left(2\int h\right)\partial_x \delta_u T \tag{14}$$

then ζ is a symmetry of (1) if and only if T is its conservation density (in the sense of [4]).

Proof. By equation (14), we have

$$P_u(\zeta) = \exp\left(2\int h\right)(\partial_x(\delta_u T)_t + k_0 \partial_x^3 \delta_u T + (6k_0 u + 4k_1 - hx)\partial_x \delta_u T + h\delta_u T). \tag{15}$$

If T is a conservation density, then there exists a function $X(t,u)$ satisfying $T_t + \partial_x X(t,u) = 0$, then $\delta_u T_t = 0$ by lemma 3.3; therefore, by lemma 3.4,

$$(\delta_u T)_t + k_0 \partial_x^3 \delta_u T + (6k_0 u + 4k_1 - hx)\partial_x \delta_u T + h\delta_u T = 0$$

so that $P_u(\zeta) = 0$, i.e. ζ is a symmetry of (1).

If ζ is a symmetry of (1), i.e. $P_u(\zeta) = 0$, then by (15) and the zero boundary conditions, we have

$$(\delta_u T)_t + k_0 \partial_x^3 \delta_u T + (6k_0 u + 4k_1 - hx)\partial_x \delta_u T + h\delta_u T = 0.$$

Furthermore by lemma 3.4 and lemma 3.3, there exist $X(t,u)$, such that

$$T_t + \partial_x X(t,u) = 0.$$

Similar to the above, we have the following theorem.

Theorem 3.2. Suppose the two functions τ, $K(t,v)$ satisfy

$$\tau = \partial_x \delta_v K \tag{16}$$

then τ is a symmetry of (2) if and only if K is its conservation density.

Combining the above facts, we have the following theorem.

Theorem 3.3. For the h-t-KdV equation, there exists an infinite number of conservation laws and there also exists a one-to-one correspondence between the symmetries and the conservation laws. This is also true for the h-t-λ-MKdV equation.

The first three conservation laws of (1) are given as follows:

$$T_1 = \exp\left(-\int h\right)u$$

$$X_1 = \exp\left(-\int h\right)\left[k_0(u_{xx} + 3u^2) + 4k_1 u - hxu\right]$$

$$T_2 = \exp\left(-3\int h\right)\tfrac{1}{2}u^2$$

$$X_2 = \exp\left(-3\int h\right)\left[k_0(uu_{xx} + 2u^3 - \tfrac{1}{2}u_x^2) + 2k_1u^2 - \tfrac{1}{2}hxu^2\right]$$

$$T_3 = \exp\left(-5\int h\right)(u^3 - \tfrac{1}{2}u_x^2)$$

$$X_3 = \exp\left(-5\int h\right)\left[k_0(3u^2u_{xx} + \tfrac{1}{2}u_{xx}^2 + \tfrac{9}{2}u^4 - 6uu_x^2 - u_xu_{xxx})\right.$$
$$\left.\bullet\ k_1(4u^3 - 2u_x^2) - hx(u^3 - \tfrac{1}{2}u_x^2)\right].$$

The first three conservation laws of (2) are given as follows:

$$K_1 = v$$

$$X_1 = k_0(v_{xx} - 2v^3) + (4k_1 + 6k_0\lambda)v - hxv$$

$$K_2 = \exp\left(-\int h\right)\tfrac{1}{2}v^2$$

$$X_2 = \exp\left(-\int h\right)\left[k_0(vv_{xx} - \tfrac{3}{2}v^4 - \tfrac{1}{2}v_x^2) + (2k_1 + 3k_0\lambda)v^2 - \tfrac{1}{2}hxv^2\right]$$

$$K_3 = \exp\left(-3\int h\right)(v^4 + v_x^2)$$

$$X_3 = \exp\left(-3\int h\right)\left[k_0(4v^3v_{xx} + 2v_xv_{xxx} - 4v^6 - 12v^2v_x^2 - v_{xx}^2)\right.$$
$$\left. + (4k_1 + 6k_0\lambda)(v^4 + v_x^2) - hx(v^4 + v_x^2)\right].$$

4. Hamiltonian structures

It is well known that both the KdV and the MKdV equations have bi-Hamiltonian structures, but for the h-t-KdV and the h-t-λ-MKdV equations there is some interesting difference, i.e. the second Hamiltonian structure of the h-t-KdV equation and the first Hamiltonian structure of the h-t-λ-MKdV equation are relatively apparant and related. Furthermore, some 'hidden' Hamiltonian structures of the h-t-KdV equation are found. They reduce to the usual bi-Hamiltonian structures when $h = 0$, i.e. when the equations are isospectral.

4.1. The second Hamiltonian structure of the h-t-KdV equation

Define the Hamiltonian function

$$H_u = \int (-\tfrac{1}{2}k_0 u^2 - 2k_1 u + \tfrac{1}{2}hxu)\,\mathrm{d}x$$

and the Poisson Bracket

$$\{\bullet, \bullet\}_u = \int \delta_u \bullet D_2 \delta_u \bullet$$

where

$$D_2 = \partial_x^3 + 2(u\partial_x + \partial_x u) = \partial_x^3 + 4u\partial_x + 2u_x.$$

Since

$$\delta_u H_u = -k_0 u - 2k_1 + \tfrac{1}{2}hx$$

$$\begin{aligned}\{u, H_u\}_u &= (\partial_x^3 + 4u\partial_x + 2u_x)(-k_0 u - 2k_1 + \tfrac{1}{2}hx)\\ &= -k_0(u_{xxx} + 6uu_x) - 4k_1 u_x + h(2u + xu_x)\end{aligned}$$

therefore

$$u_t = \{u, H_u\}_u.$$

4.2. The first Hamiltonian structure of the h-t-λ-MKdV equation

Define the Hamiltonian function

$$H_v = \int (\tfrac{1}{2}k_0(v_x^2 + v^4) - (2k_1 + 3k_0\lambda)v^2 + \tfrac{1}{2}hxv^2 + \tfrac{1}{2}hv)\,\mathrm{d}x$$

and the Poisson Bracket

$$\{\bullet, \bullet\}_u = \int \delta_u \bullet D_1 \delta_u \bullet$$

where

$$D_1 = \partial_x.$$

Since

$$\delta_v H_v = -k_0 v_{xx} + 2k_0 v^3 - (4k_1 + 6k_0\lambda)v + hxv + \tfrac{1}{2}h$$

$$\begin{aligned}\{v, H_v\}_v &= \partial_x(-k_0 v_{xx} + 2k_0 v^3 - (4k_1 + 6k_0\lambda)v + hxv + \tfrac{1}{2}h)\\ &= -k_0(v_{xxx} + 6(\lambda - v^2)v_x) - 4k_1 v_x + h(v + xv_x)\end{aligned}$$

therefore

$$v_t = \{v, H_v\}_v.$$

4.3. Connection between two Hamiltonian structures

We shall show how to obtain the second Hamiltonian structure of the *h*-*t*-KdV equation from the first Hamiltonian structure of the *h*-*t*-λ-MKdV equation via the Miura transformation.

Lemma 4.1.

$$u_t = 2h\lambda + (-\partial_x - 2v)v_t. \tag{17}$$

Lemma 4.2.

$$(-\partial_x + 2v)\delta_u H_u = \delta_v H_v + 4k_0\lambda v. \tag{18}$$

Proof.

$$\begin{aligned}(-\partial_x + 2v)\delta_u H_u &= (-\partial_x + 2v)(-k_0(\lambda - v_x - v^2) - 2k_1 + \tfrac{1}{2}hx) \\ &= -k_0 v_{xx} + 2k_0 v^3 - 2k_0\lambda v - 4k_1 v + hxv + \tfrac{1}{2}h \\ &= \delta_v H_v + 4k_0\lambda v.\end{aligned}$$

Lemma 4.3.

$$D_2 = (-\partial_x - 2v)D_1(\partial_x + 2v) + 4\lambda\partial_x. \tag{19}$$

Proof.

$$\begin{aligned}D_2 &= \partial_x^3 + 4(u - \lambda)\partial_x + 2u_x + 4\lambda\partial_x \\ &= \partial_x^3 - 4(v_x + v^2)\partial_x - 2(v_{xx} + 2vv_x) + 4\lambda\partial_x \\ &= (-\partial_x - 2v)(-\partial_x^2 + 2v\partial_x + 2v_x) + 4\lambda\partial_x \\ &= (-\partial_x - 2v)D_1(\partial_x + 2v) + 4\lambda\partial_x.\end{aligned}$$

Theorem 4.1. The second Hamiltonian structure of (1) can be derived from the first Hamiltonian structure of (2), i.e. given H_v and $\{\bullet, \bullet\}_v$ with D_1 of (2), and equations (17), (18) and (19), we can determine H_u and $\{\bullet, \bullet\}_u$ with D_2.

Proof.

(i) Determination of $\delta_u H_u$ or H_u according to (18) and H_v.
Since

$$\delta_v H_v + 4k_0\lambda v = (-\partial_x + 2v)(-k_0 u - 2k_1 + \tfrac{1}{2}hx)$$

then we can choose

$$\delta_u H_u = -k_0 u - 2k_1 + \tfrac{1}{2}hx.$$

$$H_u = \int (-\tfrac{1}{2}k_0 u^2 - 2k_1 u + \tfrac{1}{2}hxu)\,\mathrm{d}x.$$

(ii) Determination of $\{\bullet, \bullet\}_u$ according to (17), (18), (19) and $\{\bullet, \bullet\}_v$.
Since

$$\begin{aligned}u_t &= 2h\lambda + (-\partial_x - 2v)v_t \\ &= 2h\lambda + (-\partial_x - 2v)D_1((-\partial_x + 2v)\delta_u H_u - 4k_0\lambda v) \\ &= (D_2 - 4\lambda\partial_x)\delta_u H_u - (-\partial_x - 2v)D_1(4k_0\lambda v) + 2h\lambda \\ &= D_2\delta_u H_u + 4\lambda k_0 u_x - 2h\lambda + 4k_0\lambda(v_{xx} + v_x) + 2h\lambda \\ &= D_2\delta_u H_u\end{aligned}$$

therefore, we can define

$$\{\bullet, \bullet\}_u = \int \delta_u \bullet D_2 \delta_u \bullet \ \mathrm{d}x$$

so that

$$u_t = \{u, H_u\}_u.$$

4.4. Some hidden Hamiltonian structures

(i) The hidden Hamiltonian structure of $\lambda^{-\frac{1}{2}}u$.
Let

$$w = \lambda^{-\frac{1}{2}}u = \exp\left(-\int h\right)u$$

then w satisfies the equation

$$w_t + k_0(w_{xxx} + 6\lambda^{\frac{1}{2}}ww_x) + 4k_1w_x - h(w + xw_x) = 0$$

and choose

$$H = \int \left[-k_0\lambda^{\frac{1}{2}}w^3 + \tfrac{1}{2}k_0w_x^2 - 2k_1w^2 + \tfrac{1}{2}hxw^2\right] \mathrm{d}x$$

$$\{\bullet, \bullet\} = \int \delta_w \bullet D_1\delta_w \bullet \; \mathrm{d}x$$

we have

$$w_t = \{w, H\}.$$

(ii) The hidden Hamiltonian structure of $\lambda^{-\frac{3}{4}}u$.
Let

$$w = \lambda^{-\frac{3}{4}}u = \exp\left(-\frac{3}{2}\int h\right)u$$

then w satisfies the equation

$$w_t + k_0(w_{xxx} + 6\lambda^{\frac{3}{4}}ww_x) + 4k_1w_x - h(\tfrac{1}{2}w + xw_x) = 0$$

and choose

$$H = \int -\tfrac{1}{2}w^2 \, \mathrm{d}x$$

$$\{\bullet, \bullet\} = \int \delta_w \bullet D_3\delta_w \bullet \; \mathrm{d}x$$

where

$$D_3 = k_0\partial_x^3 + 2k_0\lambda^{\frac{3}{4}}(w\partial_x + \partial_x w) + 4k_1\partial_x - \tfrac{1}{2}h(x\partial_x + \partial_x x)$$

we have

$$w_t = \{w, H\}.$$

Remark. When $h \equiv 0, k_1 \equiv 0$ and $k_0 \equiv 1$, the above results clearly give the first and the second Hamiltonian structures of the standard KdV equation.

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Appendix. Proofs of Hamiltonian operators for D_1, D_2 and D_3

It has been proved that D_1, D_2 are Hamiltonian operators [9, 10]. To prove that D_3 is a Hamiltonian operator, we need only to check the Jacobi identity, i.e. the following equality holds:

$$\alpha \wedge V_{J\alpha}(J) \wedge \alpha \equiv 0 \qquad (\mathrm{mod}\, \partial_x)$$

where $J = D_3$ (see [9, 10]).

Since

$$J\alpha = k_0\alpha_{xxx} + (4k_0\lambda^{\frac{3}{4}} w + 4k_1 - hx)\alpha_x + (2k_0\lambda^{\frac{3}{4}} w_x - \tfrac{1}{2}h)\alpha$$

$$V_{J\alpha}(J) = 4k_0\lambda^{\frac{3}{4}} J\alpha\partial_x + 2k_0\lambda^{\frac{3}{4}}\partial_x(J\alpha)$$

therefore

$$\alpha \wedge V_{J\alpha}(J) \wedge \alpha = 4k_0\lambda^{\frac{3}{4}}\ \alpha \wedge J\alpha \wedge \alpha_x = 4k_0^2\lambda^{\frac{3}{4}}\ \partial_x(\alpha \wedge \alpha_{xx} \wedge \alpha_x)$$

so that D_3 is a Hamiltonian operator.

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